11.4: The Comparison Test and Limit Comparison Test.

Much like we did with the comparison test for improper integrals in 7.8, we can sometimes determine the convergence of a series $\sum_{n=1}^{\infty} a_n$ by comparing it to a known series $\sum_{n=1}^{\infty} b_n$

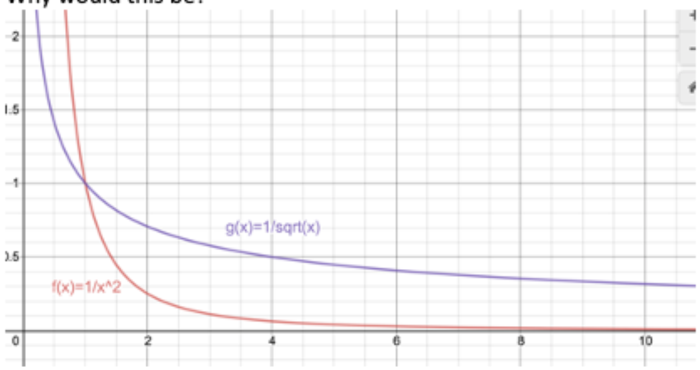
The Comparison Test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both series consisting of positive terms (or ulimately positive)

i) If $\sum_{n=1}^{\infty} b_n$ is convergent and $b_n \geq a_n$ then $\sum_{n=1}^{\infty} a_n$ is _________________________

ii) If $\sum_{n=1}^{\infty} b_n$ is divergent and b_n __________ a_n then $\sum_{n=1}^{\infty} a_n$ is divergent

Why would this be?



Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n - \frac{1}{2}}$$

Important notation detail:

We compare terms of series, not series them selves

We discuss convergence of series, not of terms of series.

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 - 5}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n + n}$$

$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4 + 4}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

useful fact: ____________

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

useful fact: ____________

$\sum_{n=1}^{\infty} \frac{1}{n^3 - n}$ ……………………… have to get creative

Limit Comparison Test

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both series consisting of positive terms (or ulimately positive) and if $c = \lim_{n\to\infty} \frac{a_n}{b_n}$, then

If C is finite and C > 0 the both series converge or both series diverge.

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 - n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 7}}$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

note:____________________________

Review thus far:

What about series with all negative terms?

What about series which are neither ultimately positive nor ultimately negative?

In particular, consider Alternating Series: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n; \quad b_n > 0$

11.5 Alternating Series Test

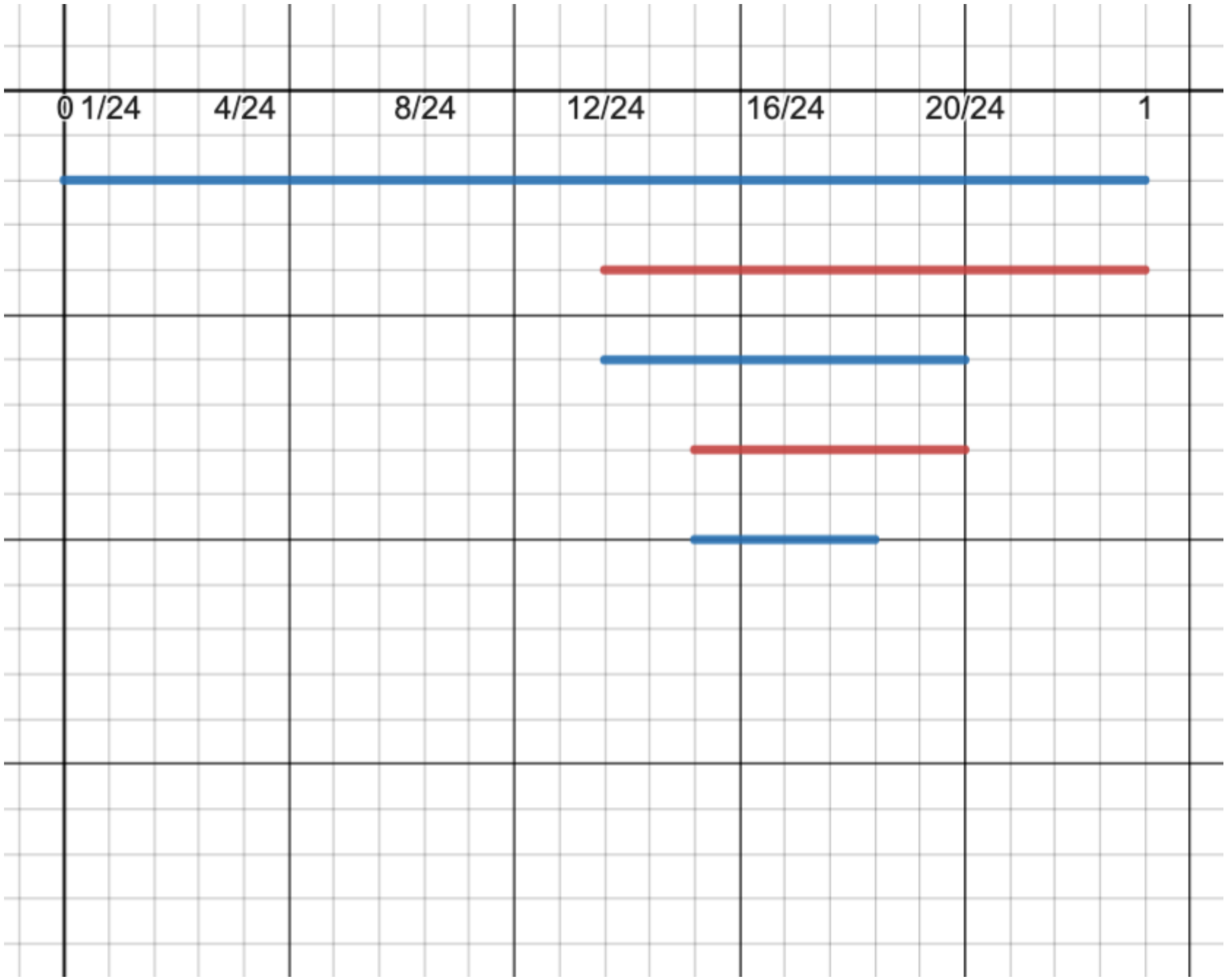
Here we examine series of the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n b_n$ or $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n; \quad b_n > 0$,

Motivating Example: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$a_1 = \_\_\_\_\_, a_2 = \_\_\_\_\_, a_3 = \_\_\_\_\_, \ldots a_n = \_\_\_\_\_, \ldots$$
$$b_1 = \_\_\_\_\_, b_2 = \_\_\_\_\_, b_3 = \_\_\_\_\_, \ldots b_n = \_\_\_\_\_, \ldots$$

Consider the sequence of partial sums:

Visualizing this graphically:


0
1/24
4/24
8/24
12/24
16/24
20/24
1

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$$

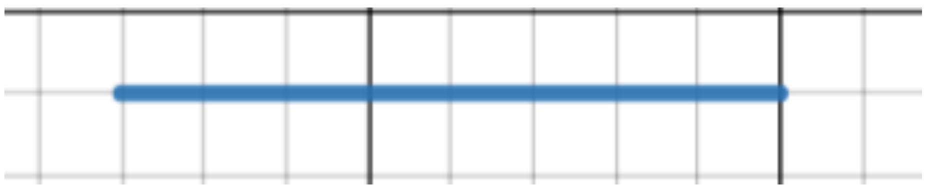
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{3^{n-1}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n} 3n}{4n-1}$$

Error Estimate for AST

In general, we have

 OR. 

Sum must be" trapped" between successive terms of the sequence of partial sums.

So the error in approximating S by Sn is

$$R_n = \left| S - \_\_\_\_\_ \right| \leq \left| S_{n+1} - S_n \right| = \_\_\_\_\_\_\_\_$$

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

(a) Estimate the sum using S_{10} $\qquad S_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \approx$

(b) What is the bound on the error if S_{10} is used to approximate S.

(c) How many terms would be needed to obtain an error < 0.05?

11.6 Absolute Convergence, Ratio Test and Root Test

Absolute Convergence:

Here will look at the relationship between the convergence of a series $\sum_{n=1}^{\infty} a_n$ and a series where we take the absolute value of each term $\sum_{n=1}^{\infty} |a_n|$.

For example:

(!) If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ = ______________________________ then $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right|$ = __________________________________ = $\sum_{n=1}^{\infty} -$

In this case, both are ____-

However,

(2) If $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ = ______________________________ then $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ = ______________________________ = _____________

In. this case

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ _______________________ while $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ = _________________________

A series $\sum_{n=1}^{\infty} a_n$ is called ______________________________________ if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If the series $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, $\sum_{n=1}^{\infty} a_n$ is called ______________________________

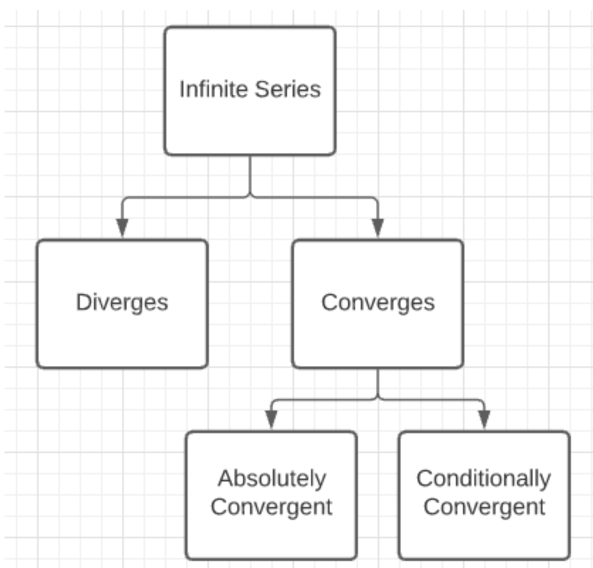
How does absolute convergence relate to "regular" convergence?

Theorem: If $\sum_{n=1}^{\infty} a_n$ is abolutley convergent then it is convergent, that is: ABSOLUTE CONVERGENCE $\Rightarrow$ CONVERGENCE

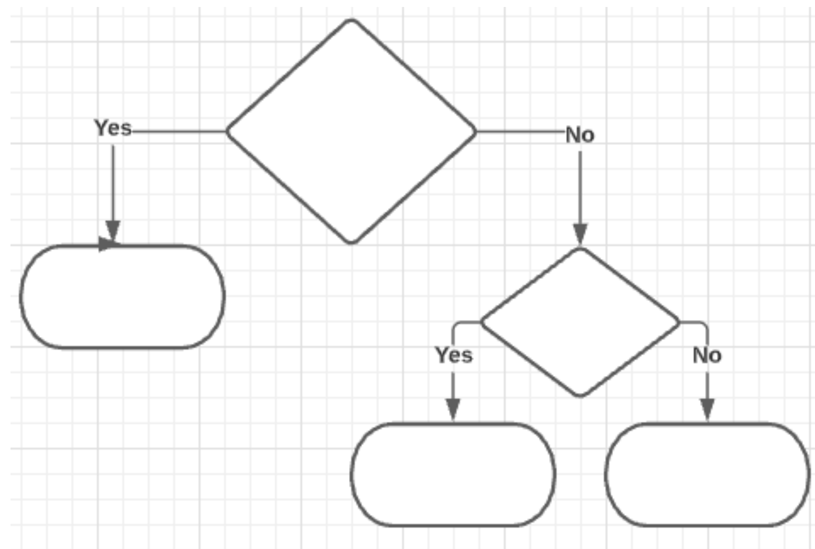
Proof:

What about converse?

Classification



Strategy



Example: From last section

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln n}$$

Ratio Test

Idea: Recall geometric series:

$$\frac{a_{n+1}}{a_n} = \_\_\_\_\_\_ \quad \text{and the geometric series converges for } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

It seems plausible that for a general series (not necessarily geometric) if $\left|\frac{a_{n+1}}{a_n}\right| < \_\_\_\_\_\_$ as $n \to \infty$ then the series ____________________

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

See proof which uses the comparison test for our series to a geometric series.

Examples:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{10^n}$$

$$\sum_{n=1}^{\infty} \frac{5}{(2n+1)!}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+3)}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$$

The Root Test

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

This is also proved by comparing to a geometric series.

Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n e^{2n}}{n^n} \qquad \sum_{n=1}^{\infty} \frac{n^3+1}{2^n}$$

11.7 Putting it all together – Strategies and Practice!

Heirachy:

Summary of Tests: